Reply to "Comment on 'Phase-difference operator' "

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In response to the objection raised by Pegg and Vaccaro [the preceding Comment] we point out that their example is not a difficulty in directly representing the phase difference by the Φ_{12} operator. In fact, we argue that it is a property common to a wide range of phase descriptions, including the Pegg-Barnett formalism.

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Pegg and Vaccaro have pointed out [1] a difficulty associated with defining the phase difference between arbitrary fields by the operator Φ_{12} introduced in Ref. [2]. Their reasoning has two steps. First, they argue that there are two ways to proceed from Φ_{12} to represent the phase difference: one is simply to define the phase of a field state in terms of the phase difference between this state and a very strong reference field in a coherent state of well-defined phase and then assume that phase differences between arbitrary fields, such as two fields in the quantum regime, can be obtained as the difference of their phases [called by them approach (a) and that coincides in this limit with the Pegg-Barnett formalism [3]], and the other one is to assume that Φ_{12} strictly represents the phase difference [called approach (b)]. Next, they claim that these two approaches are contradictory by means of a particular example, and conclude that the approach (a) is more acceptable.

In response we shall try to negate these two points independently, although we shall show that our two answers have a common origin.

We cannot agree with their first assertion since we do not think that there are two ways to proceed from Φ_{12}. The main point of our work is precisely that, due to the periodicity, *the phase difference is not simply the difference of the phases* (as it were a position or momentum difference). This assertion can be supported by the fact that the polar decomposition proposed in Ref. [2] has a unitary solution for the exponential of the phase difference [otherwise well known in the context of the SU(2) algebra [4]] contrary to the situation for the one-mode phase [5]. Moreover, this solution cannot be written as the product of phase operators for the corresponding one-mode fields. Thus it is not surprising that both approaches can lead to different results.

Despite this important point, we shall show that the two approaches are not so conflicting, at least for the particular example proposed by Pegg and Vaccaro. To do this we will follow approach (a), or equivalently, the Pegg-Barnett formalism for the phase difference. Thus we must start from the individual phases ϕ_1 and ϕ_2 of the modes involved. The Pegg-Barnett approach results in a prescription for a phase probability distribution function in terms of the Susskind-Glogower phase states. For the states considered by Pegg and Vaccaro, $|\beta\rangle = (1/\sqrt{2})(|0\rangle + |2\rangle)$ and $|\gamma\rangle = (1/\sqrt{2})(|0\rangle + \exp i\theta|1\rangle)$ they are

$$P_\beta(\phi_1) = \frac{1}{2\pi}\left[1+\cos(2\phi_1)\right],$$
$$P_\gamma(\phi_2) = \frac{1}{2\pi}\left[1+\cos(\phi_2-\theta)\right], \tag{1}$$

and then, since the two modes are considered as independent, $P_\psi(\phi_1,\phi_2) = P_\beta(\phi_1)P_\gamma(\phi_2)$. Our conclusions could be obtained directly from these expressions but we prefer to go a little further and get the probability distribution function associated with the phase difference. So, we ask for the probability distribution corresponding to the phase-sum ϕ_+ and phase-difference ϕ_- variables

$$\phi_+ = \phi_1 + \phi_2, \qquad \phi_- = \phi_1 - \phi_2, \tag{2}$$

that is,

$$\phi_1 = \frac{\phi_+ + \phi_-}{2}, \qquad \phi_2 = \frac{\phi_+ - \phi_-}{2}. \tag{3}$$

Due to the 2π-periodic character of these variables, the probability distribution corresponding to (ϕ_+,ϕ_-) cannot be obtained simply by putting (3) into $P_\psi(\phi_1,\phi_2)$. Note that in (2) it seems that the range of (ϕ_+,ϕ_-) should be a 4π-interval, or, equivalently, that (3) is not compatible with the 2π periodicity. Then, the substitution of (3) into $P_\psi(\phi_1,\phi_2)$ must be followed by a procedure casting ϕ_+ and ϕ_- into 2π ranges [6]. This is the point where the phase difference fails to be simply the difference of the phases, as discussed above. This casting can be done in many ways, but we think the clearest one starts by noting that the same mod(2π) value for (ϕ_+,ϕ_-) is obtained from two different values of (ϕ_1,ϕ_2); specifically, from (ϕ_1,ϕ_2) and $(\phi_1+\pi,\phi_2+\pi)$ [this last pair must be understood as mod(2π)]. So, to obtain the probability for the (ϕ_+,ϕ_-) values we must add the probabilities corresponding to individual phases in such a way that [taking into account the Jacobean of transformation (3)]

$$\mathcal{P}_{2\pi}(\phi_+,\phi_-) = \frac{1}{2}\left[P_\psi\left(\frac{\phi_+ + \phi_-}{2},\frac{\phi_+ - \phi_-}{2}\right) + P_\psi\left(\frac{\phi_+ + \phi_-}{2}+\pi,\frac{\phi_+ - \phi_-}{2}+\pi\right)\right], \tag{4}$$

$\mathcal{P}_{2\pi}(\phi_+,\phi_-)$ being now 2π-periodic over the ϕ_+ and ϕ_- variables. For the state $|\beta\rangle|\gamma\rangle$ considered by Pegg and Vaccaro we have

$$\mathcal{P}_{2\pi}(\phi_+,\phi_-) = \frac{1}{(2\pi)^2}\left[1+\cos(\phi_+ + \phi_-)\right]. \tag{5}$$

Here we are interested in the marginal distribution for the phase difference that can be obtained by just integrating over the phase-sum variable ϕ_+. The result for the state $|\beta\rangle|\gamma\rangle$ is

$$\mathcal{P}_{2\pi}(\phi_-) = \frac{1}{2\pi}, \tag{6}$$

which is *independent of* θ, as it is the (discrete) probability distribution function associated with the Φ_{12} operator. Then, also in the approach (a) the phase difference in this particular case is not affected by a phase shifter.

In their comment, Pegg and Vaccaro assert that this result [now common to approaches (a) and (b)] is inconsistent with the phase shifting of a field in the quantum region. We cannot agree with this, since we have just shown that the consequences on the phase difference of the action of a phase shifter are not independent in general of the two-mode field state, especially if one or both of the fields are in the quantum region. Once one is aware of this, there are no contradictions or inconsistencies in directly representing the phase difference between arbitrary fields by the Φ_{12} operator.

Recently, we have compared these two approaches for the phase properties of light propagating in a Kerr medium [7], showing that they share properties not evident at a first sight. A more detailed study of the consequences of the casting procedure will be presented elsewhere.

We think that this discussion gives a complete response to the questions raised in the preceding Comment. Nevertheless, we wish to note that this behavior for the phase difference is not exclusive of the two previous approaches. For example, we can consider the operational proposal of Noh, Fougères, and Mandel [8] based on an eight-port homodyne scheme. The state whose phase difference is to be measured is at two of the input ports, while the vacuum is at the other two. At the output ports four detectors count the photons simultaneously in each measurement, and these quadruplets of photon counts n_3, n_4, n_5, n_6 (we shall call $\{n\}$ such a set) represent the outcome of one measurement. The probability of such an outcome is

$$P_\psi(\{n\}) = |\langle\{n\}|U|\psi\rangle|^2, \tag{7}$$

where $|\{n\}\rangle$ denotes the corresponding product number state for the four output modes, and U the unitary operator performing the input-output transformation. Their approach continues identifying these probabilities with a measurement of the phase difference by means of a classical treatment of the experiment. We do not need to specify here how to do this, since we are going to see that for the state considered by Pegg and Vaccaro, all these $P_\psi(\{n\})$ are independent of θ. The point is that the input-output transformation commutes with the total photon number of the four modes involved (the device is made of beam splitters and a $\lambda/4$ retarding plate). For the example of Pegg and Vaccaro we can write the incident state $|\psi\rangle = |\beta\rangle|\gamma\rangle|0\rangle|0\rangle$ as

$$\begin{aligned}|\psi\rangle = \frac{1}{2}\,(&|0,0,0,0\rangle + \exp i\theta|0,1,0,0\rangle \\ &+ |2,0,0,0\rangle + \exp i\theta|2,1,0,0\rangle)\,,\end{aligned} \tag{8}$$

in terms of the corresponding number states $|n_1,n_2,n_{10},n_{20}\rangle$ for the input modes. Since the total photon number is conserved by the transformation we have $\langle n_3,n_4,n_5,n_6|U|n_1,n_2,n_{10},n_{20}\rangle = 0$ unless $n_3 + n_4 + n_5 + n_6 = n_1 + n_2 + n_{10} + n_{20}$. We can split the probabilities (7) for the input state (8) into four groups according to the total photon number measured. Calling $N = n_3 + n_4 + n_5 + n_6$ we have

$$P_\psi(\{n\}) = \begin{cases} \frac{1}{4}|\langle\{n\}|U|0,0,0,0\rangle|^2 & \text{if} \quad N=0 \\ \frac{1}{4}|\langle\{n\}|U|0,1,0,0\rangle|^2 & \text{if} \quad N=1 \\ \frac{1}{4}|\langle\{n\}|U|2,0,0,0\rangle|^2 & \text{if} \quad N=2 \\ \frac{1}{4}|\langle\{n\}|U|2,1,0,0\rangle|^2 & \text{if} \quad N=3 \\ 0 & \text{otherwise}\,. \end{cases} \tag{9}$$

Thus we can see that all the probabilities are independent of θ, so it will be any result for the phase difference in this approach. However, if the input state is $|\alpha\rangle|\gamma\rangle|0\rangle|0\rangle$, with $|\alpha\rangle$ a coherent state, the probabilities would depend on θ. We have argued before that this is not a contradiction.

Another example close to the previous one could be the simpler homodyne scheme considered in Ref. [2]. If we let the incident state at the two input ports be $|\beta\rangle|\gamma\rangle$, we easily get that the photon statistics at the output ports are again independent of θ, while it will depend on θ if we change $|\beta\rangle$ by a very strong reference field in a coherent state.

The same results are obtained with a phase description in terms of a quasiprobability distribution function like the Q function [9] or the Wigner function [10], once we perform the same casting procedure as in approach (a). It should be understood that these are not examples of a practical measurement scheme for the Φ_{12} operator. Our idea is just to show that the behavior noted by Pegg and Vaccaro is, in fact, quite general.

In conclusion, if a phase measurement must be thought of as a measurement of the phase difference, the results must depend in general on the reference state. Such a dependence cannot be accounted for simply by a difference of phases of these states with a coherent state of sufficient intensity, due to the periodic character of this variable. This is precisely the central idea for the definition of Φ_{12} in Ref. [2] that naturally reflects these facts by means of a phase-difference operator.

In any case, we wish to point out that the previous comments of Pegg and Vaccaro are appropriate and valuable since they deal with a subtle point of this problem.

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